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A NEW THEORY HOWELL'S SPACE AND FRACTIONAL HARTLEY TRANSFORM

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Abstract:

This paper introduced the notation of fractional Hartley transform on Howell's space as Howell himself had extended Fourier transform in the series of paper the main goal are the introduction of the basic space of test function. Some fundamental results arediscussed in the context of fractional Hartley transform.

Key words: Fractional Fourier transforms Howell's space, Fractional Hartley transform

AMS subject code: 46F12 and 44

1. Introduction:

A new theory of Fourier analysis presented in a short series of papers [1-3] supporting the Fourier transform of the function on R^n. This theory is similar to the theory for tempered distribution and to some extent, can be viewed as an extension of the Fourier theory except that there will be no need to restrict attention to function of polynomial growth. Any locally integrable function of exponential growth on R^n will be Fourier transformable, so can be therefore fractional Fourier transformable. Motivated by thiswe are extending this theory to support fractional Hartley transform[6].

In this paper we have introduced the notation of fractional Hartley transform on Howell's space as a Howell himself had extended Fourier transform in the series of paper [1-3].The main goals are the introduction of the basic space of test function in section 2. Some fundamental results are discussed in the context of fractional Hartley transform in section 3.

2 The Testing Function Spaces G and G^C :

Purpose of this section is to provide framework for defining fractional Hartley transform on Howell's space. Two spaces of functions, G and G^C conceived by Howell will be of special importance.

A function $\phi : C^n \to C$ is an element of G if and only if both the following hold.

1] ϕ is an analytical function of each complex variable

2] For every $v > 0$

$$Sup\left\{ e^{v\rho(z)} \left| \phi(z) \right| : z \in \beta_v \right\} < \infty ,$$

$$\because \rho(z) = \sum_{k=1}^{n} \left| \mathrm{Re}[z_k] \right|$$

and β_v will denote the subset of C^n.

A function $f : C^n \to C$ is an element of G^C if and only if both the following hold.

1] f is an analytical function of each complex variable.

2] For any $v > 0$ there is a corresponding $\gamma \geq 0$ such that

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$$Sup\left\{e^{-v\rho(z)}\left|f(z)\right|: z\in\beta_v\right\}<\infty$$

G with the suitable topology will serve as the space of test functions. This is ascertained by restraining the parameter α involved in the kernel of fractional Hartley transform to $(0,1]$.

The following facts will be obvious from the above definition

1] G, G^C are both linear spaces closed under multiplication

2] $G\subset G^C$

3] G contains all the functions of the form $\phi=\Theta_{\xi}\eta_v$, $v>0$ and $\xi\in C^n$, where Θ_{ξ} will denote the translation of f

4] If $\phi\in G$ and $\xi,\zeta\in C^n$ then $\rho_{i\xi}\,\Theta_{\zeta}\,\phi$ and therefore $K_\alpha(t,s)\Theta_{\zeta}\,\phi$ for $o\le\alpha\le1$ are also in G and its restriction to IR^n is in $L^{'}(IR^n)$, where ρ will denote the seminorm on C^n.

It is clear that $K_\alpha(t,s)\in G$, $o\le\alpha\le1$, $v>0$.

For $Sup\left\{e^{v\rho(s)}\left|K_\alpha(t,s)\right|: s\in\beta_v\right\}=Sup\left\{e^{v\rho(s)}\left|\frac{1}{\sqrt{(2\pi\sin\alpha)}}\right|: s\in\beta_v\right\}<\infty$.

It thus follows the assertion made at (4).

5] G^C contains all polynomial functions on C^n.

6] Any exponential functions ρ_ξ with $\xi\in C^n$ is an element of G^C.

The topology on G is defined by the set of norms $\left\{\|\ \|_v : v>0\right\}$

where $\|\phi\|_v=Sup\left\{e^{v\rho(s)}\left|\phi(s)\right|: s\in\beta_v\right\}$.

Any function ϕ which is analytical in each variable, is clearly in G if and only if $\|\phi\|_v$ is finite for every $v>0$.

It is also obvious that if $v>\mu$ then $\|\phi\|_v\le\|\phi\|_\mu$.

Further the function $g(s)=\phi(s)\sqrt{\frac{1-i\cot\phi}{2\pi}}\,e^{i\frac{s^2}{2}\cot\phi}$ involved in the present analysis is clearly in G for $\phi\in G$, $v>0$ and $o\le\alpha\le1$

$$Sup\left\{e^{v\rho(s)}\left|g(s)\right|: s\in\beta_v\right\}=Sup\left\{e^{v\rho(s)}\left|\phi(s)\right|\sqrt{\frac{1-i\cot\phi}{2\pi}}\,e^{i\frac{s^2}{2}\cot\phi}: s\in\beta_v\right\}$$

$$=Sup\left\{e^{v\rho(s)}\left|\phi(s)\right|\left|\sqrt{\frac{1-i\cot\phi}{2\pi}}\,e^{i\frac{s^2}{2}\cot\phi}\right|: s\in\beta_v\right\}$$

$$=\sqrt{\frac{1-i\cot\phi}{2\pi}}\,e^{i\frac{s^2}{2}\cot\phi}\,Sup\left\{e^{v\rho(s)}\left|\phi(s)\right|: s\in\beta_v\right\}<\infty.$$

2.1 Howell's Space:

The Kenneth B. Howell in 1991, developed a theory entitled "A new theory of Fourier analysis", presented in an elegant series of papers [1]-[3] supporting the Fourier transformation of the functions on

IR^n. His work is the development of a theory, generalizing the well known theory of tempered distributions, which naturally supports fractionalization of all exponentially bounded functions.

There will be no need to restrict attention to function of polynomial growth as is generally the case with former. Any locally integrable function of exponential growth is Fourier transformable. So can be therefore fractional Fourier transformable. Motivated by this we are extending this theory to support fractional Hartley transform.

The notation and the terminologies in this work follow those of [1]-[3]. We list some notation from Howell [1], which will be used prominently in the paper.

The translation and scaling operators $\Theta_\xi \dfrac{f}{z} = f(z-\xi)$.

Let $f : C^n \to C$. For each $\xi \in C^n$, Θ_ξ will denote the translation of f and ξ, given by $\Theta_\xi \dfrac{f}{z} = f(z-\xi) = \Theta_z$.

For each $v \in C$, $\sum_v {f}/{z} = f(vz) = \sum_v f$.

The ''Rho'' seminorm: The symbol ρ will denote the seminorm on C^n given by $\rho(z) = \sum_{k=1}^{n} |\ \mathrm{Re}[z_k]|$

.

The Exponential function: Let $w = C^n$, where convenient e_w will denote the function given by $e_w(z) = e^{wz}$.

The Normal Gaussian: Let $v > 0$ where convenient, η_v will denote the

$$\eta_v(z) = \left(\frac{v}{\sqrt{2\pi}}\right)^n \exp\left[-\frac{(vz)^2}{2}\right].$$

3 Some Lemmas:

3.1 Lemma:(Scaling and the fractional Hartley transform on G)

Let β be any non zero real number and let ϕ be any elements in G.

Then $$H^\alpha \sum_\beta \{\phi(t)\} = e^{\frac{i}{2}s^2 \cot\phi (1-\frac{1}{\beta^2})} \beta^{-1} \sum_{\beta^{-1}} H^\alpha \left[e^{\frac{i}{2}(\frac{1}{\beta^2}-1)t^2 \cot\phi} \phi(t)\right](s).$$

{In particular, this lemma implies that H^α and $\sum_{-1}$ commutes on G}

Proof: Let β be any non zero real number and let $\phi \in G$, we have

$$H^\alpha \left\{ \sum_\beta [\phi(t)] \right\} = \int_{IR^n} \sum_\beta \phi(t) K_\alpha(t,s)\, d\Lambda_t$$

$$= \sqrt{\frac{1 - i\cot\phi}{2\pi}}\, e^{i\frac{s^2}{2}\cot\phi} \int_{IR^n} \sum_\beta \phi(t)\, e^{i\frac{t^2}{2}\cot\phi} \left[\cos(\csc\phi .st) - ie^{i\phi}\sin(\csc\phi .st)\right] d\Lambda_t$$

$\because \sum_\beta [\phi(t)] = \phi(t\beta)$ using[1]

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$$=\sqrt{\frac{1-i\cot\phi}{2\pi}}\, e^{i\frac{s^2}{2}\cot\phi}\int_{IR^n}\phi(t\beta)\, e^{i\frac{t^2}{2}\cot\phi}\left[\cos(\csc\phi.st)-ie^{i\phi}\sin(\csc\phi.st)\right]d\Lambda_t\,.$$

Put $t\beta = T \;\therefore t=\beta^{-1}T \;\; d\Lambda_t=\beta^{-1}d\Lambda_T$

$$=\sqrt{\frac{1-i\cot\phi}{2\pi}}\, e^{i\frac{s^2}{2}\cot\phi}\int_{IR^n}\phi(T)\, e^{\frac{i}{2}\left(\frac{T}{\beta}\right)^2\cot\phi}\left[\cos\left(\csc\phi.s\left(\frac{T}{\beta}\right)\right)-ie^{i\phi}\sin\left(\csc\phi.s\left(\frac{T}{\beta}\right)\right)\right]\beta^{-1}d\Lambda_T$$

$$=\sqrt{\frac{1-i\cot\phi}{2\pi}}\, e^{\frac{i}{2}\frac{s^2}{\beta^2}\cot\phi}\, e^{-\frac{i}{2}\frac{s^2}{\beta^2}\cot\phi+\frac{i}{2}s^2\cot\phi}\,\frac{1}{\beta}\int_{IR^n}\phi(T)\, e^{\frac{i}{2}T^2\cot\phi}\, e^{-\frac{i}{2}T^2\cot\phi+\frac{i}{2}\frac{T^2}{\beta^2}\cot\phi}$$

$$\left[\cos\left(\csc\phi.\left(\frac{s}{\beta}\right)T\right)-ie^{i\phi}\sin\left(\csc\phi.\left(\frac{s}{\beta}\right)T\right)\right]d\Lambda_T$$

$$=\frac{1}{\beta}e^{\frac{i}{2}\left(1-\frac{1}{\beta^2}\right)s^2\cot^2\phi}\sqrt{\frac{1-i\cot\phi}{2\pi}}\, e^{\frac{i}{2}\frac{s^2}{\beta^2}\cot\phi}\int_{IR^n}\phi(T)\, e^{\frac{i}{2}T^2\cot\phi}\, e^{\frac{i}{2}\left(\frac{1}{\beta^2}-1\right)T^2\cot\phi}$$

$$\left[\cos\left(\csc\phi.\left(\frac{s}{\beta}\right)T\right)-ie^{i\phi}\sin\left(\csc\phi.\left(\frac{s}{\beta}\right)T\right)\right]d\Lambda_T$$

$$=\frac{1}{\beta}e^{\frac{i}{2}s^2\cot^2\phi}\, e^{-\frac{i}{2}\frac{s^2}{\beta^2}\cot\phi}.H^\alpha\left\{e^{\frac{i}{2}\left(\frac{1}{\beta^2}-1\right)T^2\cot\phi}\phi(T)\right\}\left(\frac{s}{\beta}\right)$$

$$=\beta^{-1}e^{\frac{i}{2}s^2\cot^2\phi\left(1-\frac{1}{\beta^2}\right)}\sum\nolimits_{\beta^{-1}}.H^\alpha\left\{e^{\frac{i}{2}\left(\frac{1}{\beta^2}-1\right)T^2\cot\phi}\phi(T)\right\}(s)$$

$$=\beta^{-1}e^{\frac{i}{2}s^2\cot^2\phi\left(1-\frac{1}{\beta^2}\right)}\sum\nolimits_{\beta^{-1}}.H^\alpha\left\{e^{\frac{i}{2}\left(\frac{1}{\beta^2}-1\right)t^2\cot\phi}\phi(t)\right\}(s).$$

3.2 Lemma: (Translation and the fractional Hartley transform on G):

Let $\phi\in G$ then for any ξ and ζ in C^n then

$$H^\alpha\left\{e_{i\xi}\Theta_\xi\phi(t)\right\}=\cos(\csc\phi.s\zeta)e^{i\frac{\zeta^2}{2}\cot\phi+i\xi\zeta}H^\alpha\left\{\phi(t)e_{i(\zeta\cot\phi+\xi)}\right\}(s)-\sin(\csc\phi.s\zeta)$$

$$e^{i\frac{\zeta^2}{2}\cot\phi+i\xi\zeta}\left\{i\cot\phi H^\alpha\left\{\phi(T)e_{i(\zeta\cot\phi+\xi)}\right\}(s)-\sin\phi H^\alpha\left\{\phi(-T)e_{i(\zeta\cot\phi+\xi)}\right\}(s)\right\}.$$

Proof: Let $\phi\in G$ then for any ξ and ζ in C^n

$$H^{\alpha}\left\{e_{i\xi}\Theta_{\xi}\phi(t)\right\}$$

$$=\sqrt{\frac{1-i\cot\phi}{2\pi}}e^{i\frac{s^2}{2}\cot\phi}\int_{-\infty}^{\infty}e^{i\frac{t^2}{2}\cot\phi}\left[\cos(\csc\phi.st)-ie^{i\phi}\sin(\csc\phi.st)\right]e^{i\xi t}\,\phi(t-\xi)\,d\Lambda_t$$

put $t-\zeta=T$, $t=T+\zeta$, $d\Lambda_t=d\Lambda_T$

$$=\sqrt{\frac{1-i\cot\phi}{2\pi}}\,e^{i\frac{s^2}{2}\cot\phi}\int_{-\infty}^{\infty}e^{\frac{i}{2}(T+\zeta)^2\cot\phi}\left[\cos(\csc\phi.s(T+\zeta))-ie^{i\phi}\sin(\csc\phi.s(T+\zeta))\right]$$

$$e^{i\xi(T+\zeta)}\,\phi(T)\,d\Lambda_T$$

$$=\sqrt{\frac{1-i\cot\phi}{2\pi}}e^{i\frac{s^2}{2}\cot\phi}\int_{-\infty}^{\infty}e^{\frac{i}{2}(T^2+2T\zeta+\zeta^2)\cot\phi}\left[\cos(\csc\phi.sT+\csc\phi.s\zeta)-ie^{i\phi}\sin(\csc\phi.sT\right.$$

$$\left.+\csc\phi,s\zeta)\right]e^{i\xi(T+\zeta)}\,\phi(T)\,d\Lambda_T$$

$$=\sqrt{\frac{1-i\cot\phi}{2\pi}}\,e^{i\frac{s^2}{2}\cot\phi}\,e^{i\frac{\zeta^2}{2}\cot\phi}\int_{-\infty}^{\infty}e^{i\frac{T^2}{2}\cot\phi}\,e^{iT\zeta\cot\phi}e^{i\xi(T+\zeta)}\left[\cos(\csc\phi.s\zeta)\left(\cos(\csc\phi.sT)\right.\right.$$

$$\left.\left.-ie^{i\phi}\sin(\csc\phi.sT)\right)-\sin(\csc\phi.s\zeta)\left(\sin(\csc\phi.sT)+ie^{i\phi}\cos(\csc\phi.sT)\right)\right]\phi(T)\,d\Lambda_T$$

$$=\sqrt{\frac{1-i\cot\phi}{2\pi}}e^{i\frac{s^2}{2}\cot\phi}\,e^{i\frac{\zeta^2}{2}\cot\phi}\,\cos(\csc\phi.s\zeta)\int_{-\infty}^{\infty}e^{i\frac{T^2}{2}\cot\phi}\,e^{iT\zeta\cot\phi}e^{i\xi(T+\zeta)}\left(\cos(\csc\phi.sT)\right.$$

$$\left.-ie^{i\phi}\sin(\csc\phi.sT)\right)\phi(T)\,d\Lambda_T-\sqrt{\frac{1-i\cot\phi}{2\pi}}e^{i\frac{s^2}{2}\cot\phi}\,e^{i\frac{\zeta^2}{2}\cot\phi}\,\sin(\csc\phi.s\zeta)\int_{-\infty}^{\infty}e^{i\frac{T^2}{2}\cot\phi}$$

$$e^{iT\zeta\cot\phi}\,e^{i\xi(T+\zeta)}\left(\sin(\csc\phi.sT)+ie^{i\phi}\cos(\csc\phi.sT)\right)\phi(T)\,d\Lambda_T$$

$$=\cos(\csc\phi.s\zeta)e^{i\frac{\zeta^2}{2}\cot\phi}\sqrt{\frac{1-i\cot\phi}{2\pi}}e^{i\frac{s^2}{2}\cot\phi}\int_{-\infty}^{\infty}e^{i\frac{T^2}{2}\cot\phi}\,e^{iT(\zeta\cot\phi+\xi)}e^{i\xi\zeta}\left(\cos(\csc\phi.sT)\right.$$

$$\left.-ie^{i\phi}\sin(\csc\phi.sT)\right)\phi(T)\,d\Lambda_T-\sin(\csc\phi.s\zeta)e^{i\frac{\zeta^2}{2}\cot\phi}\sqrt{\frac{1-i\cot\phi}{2\pi}}e^{i\frac{s^2}{2}\cot\phi}\int_{-\infty}^{\infty}e^{i\frac{T^2}{2}\cot\phi}$$

$$e^{iT(\zeta\cot\phi+\xi)}\,e^{i\xi\zeta}\left(\sin(\csc\phi.sT)+ie^{i\phi}\cos(\csc\phi.sT)\right)\phi(T)\,d\Lambda_T$$

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$$= \cos(\csc\phi.s\zeta)\, e^{i\frac{\zeta^2}{2}\cot\phi} e^{i\xi\zeta} H^\alpha\left\{\phi(T)e^{iT(\zeta\cot\phi+\xi)}\right\}(s) - \sin(\csc\phi.s\zeta)\, e^{i\frac{\zeta^2}{2}\cot\phi} e^{i\xi\zeta}$$

$$\left\{i\cot\phi.H^\alpha\left\{\phi(T)e^{iT(\zeta\cot\phi+\xi)}\right\}(s) - \sin\phi H^\alpha\left\{\phi(-T)e^{iT(\zeta\cot\phi+\xi)}\right\}(s)\right\}$$

$$= \cos(\csc\phi.s\zeta)\, e^{i\frac{\zeta^2}{2}\cot\phi+i\xi\zeta} H^\alpha\left\{\phi(t)e_{i(\zeta\cot\phi+\xi)}\right\}(s) - \sin(\csc\phi.s\zeta)$$

$$e^{i\frac{\zeta^2}{2}\cot\phi+i\xi\zeta}\left\{i\cot\phi H^\alpha\left\{\phi(T)e_{i(\zeta\cot\phi+\xi)}\right\}(s) - \sin\phi H^\alpha\left\{\phi(-T)e_{i(\zeta\cot\phi+\xi)}\right\}(s)\right\}.$$

3.3 Lemma: (The Normal Gaussian)

Let $v > 0$ where convenient, η will denote the function given by

$$\eta_v(z) = \left(\frac{v}{\sqrt{2\pi}}\right)^n \exp\left[\frac{-(vz)^2}{2}\right]$$

then (i) $H^\alpha\{\eta_v\} = \dfrac{1}{2}\left(\dfrac{V}{\sqrt{2\pi}}\right)^n \sqrt{\dfrac{1-i\cot\phi}{(\cot\phi+iV^2)}}\; e^{i\frac{s^2}{2}\left(\cot\phi+\frac{\csc^2\phi}{\cot\phi+iV^2}\right)}$.

Proof:

$$H^\alpha\{\eta_v\} = \sqrt{\frac{1-i\cot\phi}{2\pi}}\, e^{i\frac{s^2}{2}\cot\phi} \int_{IR^n} e^{i\frac{t^2}{2}\cot\phi}\left[\cos(\csc\phi.st) - ie^{i\phi}\sin(\csc\phi.st)\right]\eta_v(t)\, d\Lambda_t$$

$$H^\alpha\{\eta_v\} = C_{s,\phi}\int_{IR^n} e^{i\frac{t^2}{2}\cot\phi}\left[\cos(\csc\phi.st) - ie^{i\phi}\sin(\csc\phi.st)\right]\left[\left(\frac{V}{\sqrt{2\pi}}\right)^n \exp\left(\frac{-(Vt)^2}{2}\right)\right] d\Lambda_t$$

$$= C_{s,\phi}\int_{IR^n}\left(\frac{V}{\sqrt{2\pi}}\right)^n\left[\cos(\csc\phi.st) - ie^{i\phi}\sin(\csc\phi.st)\right]\exp\left(i\frac{t^2}{2}\cot\phi - \frac{(Vt)^2}{2}\right) d\Lambda_t$$

$$= C_{s,\phi}\left(\frac{V}{\sqrt{2\pi}}\right)^n\int_{IR^n}\left[\cos(\csc\phi.st) - ie^{i\phi}\sin(\csc\phi.st)\right]\exp\left(it^2\left(\frac{\cot\phi - \frac{V^2}{i}}{2}\right)\right) d\Lambda_t$$

put $\cot\phi - \dfrac{V^2}{i} = u^2$ or $\cot\phi + iV^2 = u^2$

$$= C_{s,\phi}\left(\frac{V}{\sqrt{2\pi}}\right)^n\int_{IR^n}\left[\cos(\csc\phi.st) - ie^{i\phi}\sin(\csc\phi.st)\right]\exp\left(\frac{it^2u^2}{2}\right) d\Lambda_t$$

$$= \left(\frac{V}{\sqrt{2\pi}}\right)^n C_{s,\phi}\left[\int_{IR^n} e^{\frac{iu^2t^2}{2}}\cos(\csc\phi.st)d\Lambda_t - ie^{i\phi}\int_{IR^n} e^{\frac{iu^2t^2}{2}}\sin(\csc\phi.st)d\Lambda_t\right]$$

$$= \left(\frac{V}{\sqrt{2\pi}}\right)^n C_{s,\phi}\int_{IR^n} e^{\frac{iu^2t^2}{2}}\cos(\csc\phi.st)\,d\Lambda_t - ie^{i\phi}\left(\frac{V}{\sqrt{2\pi}}\right)^n C_{s,\phi}\int_{IR^n} e^{\frac{iu^2t^2}{2}}\sin(\csc\phi.st)\,d\Lambda_t$$

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$$=\left(\frac{V}{\sqrt{2\pi}}\right)^n C_{s,\phi}.\frac{1}{2}\sqrt{\frac{2\pi}{u^2}}e^{\frac{i\csc^2\phi.s^2 2}{4u^2}}+0\,.$$

Using $\int_0^\infty e^{iax^2}\cos(bx)dx=\frac{1}{2}\sqrt{\frac{\pi}{a}}\,e^{\frac{ib^2}{4a}}$

and $\int_0^\infty e^{iax^2}\sin(bx)dx=0$

$$=\left(\frac{V}{\sqrt{2\pi}}\right)^n C_{s,\phi}.\sqrt{\frac{\pi}{2u^2}}e^{\frac{i\csc^2\phi.s^2}{2u^2}}$$

$$=\left(\frac{V}{\sqrt{2\pi}}\right)^n\sqrt{\frac{1-i\cot\phi}{2\pi}}\;e^{i\frac{s^2}{2}\cot\phi}.\sqrt{\frac{\pi}{2(\cot\phi+iV^2)}}\;e^{i\frac{s^2}{2}\frac{\csc^2\phi}{\cot\phi+iV^2}}$$

$$=\left(\frac{V}{\sqrt{2\pi}}\right)^n\sqrt{\frac{1-i\cot\phi}{2\pi}\times\frac{\pi}{2(\cot\phi+iV^2)}}\;\;e^{i\frac{s^2}{2}\cot\phi\;+\;i\frac{s^2}{2}\frac{\csc^2\phi}{\cot\phi+iV^2}}.$$

$$=\frac{1}{2}\left(\frac{V}{\sqrt{2\pi}}\right)^n\sqrt{\frac{1-i\cot\phi}{(\cot\phi+iV^2)}}\;\;e^{i\frac{s^2}{2}\left(\cot\phi+\frac{\csc^2\phi}{\cot\phi+iV^2}\right)}.$$

3.4 Lemma: If $f,g\in G$ then for restrictions to IR^n of the product $gH^\alpha\{f(t)\}$and $fH^\alpha\{g(t)\}$ are in $L^{'}(IR^n)$ and

$$\int_{IR^n} g\,H^\alpha\{f(t)\}d\Lambda=\int_{IR^n} f\,H^\alpha\{g(t)\}d\Lambda\,.$$

Proof: Let $f,g\in G$, let us denotes $\sqrt{\frac{1-i\cot\phi}{2\pi}}\;e^{i\frac{s^2}{2}\cot\phi}=C_{s,\phi}$

$$\therefore\, g\,H^\alpha\{\,f(t)\,\}=g\,C_{s,\phi}\int_{IR^n} e^{i\frac{t^2}{2}\cot\phi}\left[\cos(\csc\phi.st)-ie^{i\phi}\sin(\csc\phi.st)\,\right]f(t)\,d\Lambda_t$$

and $f\,H^\alpha\{\,g(t)\,\}=f\,C_{s,\phi}\int_{IR^n} e^{i\frac{t^2}{2}\cot\phi}\left[\cos(\csc\phi.st)-ie^{i\phi}\sin(\csc\phi.st)\,\right]g(t)\,d\Lambda_t$.

Obviously, the restrictions to IR^n of the product $gH^\alpha\{f\}$and $fH^\alpha\{g\}$ are in $L^{'}(IR^n)$ by Lemma 5.5 [2]. Also we have

$$\int_{IR^n} H^\alpha\{f(t)\}g(t)\,d\Lambda_t=\int_{IR^n} C_{s,\phi}\int_{IR^n} e^{i\frac{t^2}{2}\cot\phi}\left[\cos(\csc\phi.st)-ie^{i\phi}\sin(\csc\phi.st)\,\right]$$

$$f(t)\,d\Lambda_t\,g(t)\,d\Lambda_t$$

$$=\int_{IR^n} f(t)\left[C_{s,\phi}\int_{IR^n} e^{i\frac{t^2}{2}\cot\phi}\left[\cos(\csc\phi.st)-ie^{i\phi}\sin(\csc\phi.st)\,\right]g(t)\,d\Lambda_t\right]d\Lambda_t$$

$$\because \int_{IR^n} F[\phi]\psi\, d\Lambda = \int_{IR^n} \phi F[\psi]\, d\Lambda$$

$$= \int_{IR^n} f(t) H^{\alpha}\{g(t)\} d\Lambda_t \text{ or}$$

$$= \int_{IR^n} f H^{\alpha}\{g\} d\Lambda .$$

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