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LRS Bianchi type- Ifractional holographic dark energy inBrans-Dicke theory of gravitation in Lyra geometry

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Abstract

In this paper, we have discussed the exact solution of Einstein's field equations for LRS Bianchi type- I cosmological model in the framework of the scalar-tensor theory of gravitation given by Brans-Dickeand Lyra geometry. In order to obtain a determinant solution, special law of variation for Hubble's parameter proposed by Berman (1983) has been considered. The relationship between holographic dark energy model with the quintessence dark energy has been established. Phantom potential and dynamics of the Quintessence scalar field are reconstructed, which describes the accelerated phase of the expanding universe. Some physical and geometrical properties of the model are also discussed.

Keywords: Cosmological model; LRS Bianchi type- I; Brans-Dicketheory.

1. Introduction

The recent observational studies have given evidence for the accelerated expansion of the universe[1-5] (Perlmutter et al. 1999 [1]; Reiss et al. 1998 [2]; Bennett et al. 2003 [3]; Spergel et al. 2003 [4]; Tegmark et al. 2004 [5]), suggesting the presence of a mysterious form of energy that drives this acceleration. This enigmatic force is commonly referred to as "dark energy." Observations indicate that over 70% of the universe is composed of dark energy, which is responsible for the negative pressure that accelerates the cosmic expansion, while the remaining 30% consists of matter, most of which is dark matter-non-baryonic matter that does not emit or interact with electromagnetic radiation. There have been numerous other dark energy models proposed, including quintessence [6], phantom [7], quintom [8], tachyon [9], ghost [10], K-essence [11], phantom [12], Chaplygin gas [13], polytropic gas [14] and holographic dark energy (HDE) [15] and many more to explain the accelerated expansion of the universe. The current understanding of the universe posits that it is predominantly made up of cosmic fluids consisting of dark matter and dark energy, evolving independently.

A variety of cosmological models have been studied to understand the behavior of dark energy and the dynamics of the universe within the framework of modified gravitational theories. Among these, the Self-Creation Theory (SCT) of gravitation, introduced by Barber [16], and Lyra Geometry, an extension of Riemannian geometry proposed by Lyra [17], have been extensively used in cosmological investigations.

In SCT, a variable gravitational constant G is considered, which influences the evolution of the universe. Lyra geometry modifies the standard Riemannian geometry by introducing a gauge function, and as a result, the gravitational dynamics and the structure of spacetime differ from those predicted by general relativity (GR). This framework opens up new possibilities for modeling the evolution of the universe and the role of dark energy within it.

Several studies have been dedicated to exploring cosmological models within these frameworks. For instance, , Mahanta (2014) [18] studied locally rotationally symmetric Bianchi type I cosmological model. In the gravitational theory based on Lyra geometry and in the presence of a bulk viscous fluid, LRS Bianchi type I cosmological models were studied by Pradhan and Pandey (2003) [19] and Kandalkar and Samdurkar (2015) [20]. while Pradhan et al. (2007b) [21] presented a new class of LRS Bianchi type-I cosmological model in the presence of bulk viscous fluid with variable deceleration parameter in the general relativity theory. Hegazy and Rahaman (2019b) [22] studied Bianchi type $V\, I_0$ cosmological model in the second self-creation theory in general relativity and in Lyra geometry. Pawar and Solanke (2014) [23] studied magnetized anisotropic dark energy models in Barber's second self-creation theory. Interacting two-fluid viscous dark energy models in self creation cosmology was given by Chirde and Shekh (2014) [24]. Kaluza-Klein cosmological model with bulk viscosity in Barber's second self-creation cosmology was given by Kumar and Reddy (2015) [25] Reddy and Naidu (2009) [26] studied Kaluza-Klein cosmological models in

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selfcreation cosmology. Evolution of spatially homogeneous and isotropic FRW cosmological model with bulk-viscosity in self-creation theory of gravitation was analyzed by Katore et al. (2010) [27]. Samanta and Mishra (2017) [28] studied Anisotropic Cosmological Model in Presence of Holographic Dark Energy and Quintessence

The concept of Fractional Holographic Dark Energy (FHDE) was proposed by Trivedi et al. (29) and explores a modification of the standard Holographic Dark Energy (HDE) model by using fractional calculus. This modification leads to a power-law relationship between entropy and area, which is an important feature of the theory. The entropy is expressed as:

$$S_h = CA^{\frac{2+\gamma}{2\gamma}} \tag{1}$$

where γ is a parameter that modifies the entropy-area relation, resembling the Barrow and Tsallis entropies. By applying the holographic inequality, a relationship between the cosmological parameters is derived:

$$\Lambda^3 L^3 \leq \left(CA^{\frac{2+\gamma}{2\gamma}}\right)^{\frac{3}{4}} \tag{2}$$

and the energy density of FHDE is given by:

$$\rho_f = \gamma L^{\frac{2-3\gamma}{\gamma}} \tag{3}$$

where γ is a constant. In the limit where $\gamma = 2$, this gives us the standard HDE energy density. The proposed FHDE energy density is then written as:

$$\rho_f = 3c^2 L^{\frac{2-3\alpha}{\alpha}} \tag{4}$$

where c is a free dimensionless $O(1)$ parameter say arbitrary parameter, for the remaining part of the paper we will consider $c^2 = M_p^2 = 1$.

With this in mind, the definition (5) with the Hubble Horizon cutoff $L \to H^{-1}$ gives

$$\rho_f = 3c^2 H^{\frac{3\gamma-2}{\gamma}} \tag{5}$$

here we analyze the cosmological evolution at late times in the framework of the FHDE model using the Granda–Oliveros (G–O) cutoff.

$$L_{\mathrm{R}} = (\alpha H^2 + \beta \dot{H})^{-1/2} \tag{6}$$

where α and β are arbitrary dimensionless parameters. Here, Granda-Oliveros IR cutoff, the FHDE density from Eq.(6) comes out to be

$$\rho_f = 3(\alpha H^2 + \beta \dot{H})^{\frac{3\gamma-2}{2\gamma}} \tag{7}$$

The above discussion and investigations, we consider in this paper the fractional holographic dark energy model in hypersurface homogeneous spacetime within the framework of the BD scalar-tensor theory of gravitation. This work is organized as follows: In Sect. 2, we derive the SB field equations with the help of a hypersurface homogeneous spacetime metric in the presence of two minimallyA interacting fields: dark matter and holographic dark energy. Sect. 3 is devoted to the solution of BD field equations with the help of a special law of variation for Hubble's parameter proposed by Berman [30] and using physically relevant conditions. In Sect. 4, physical and kinematical parameters of the model are also computed and discussed. The last section contains some concluding remarks.

2. The Metric and Field Equations

We consider the spatially homogeneous LRS Bianchi type- I space-time as

$$ds^2 = -dt^2 + A^2(t)dx^2 + B^2(t)[dy^2 + dz^2], \tag{8}$$

where A, B are functions of cosmic time t only.

The Einstein's Brans-Dicke field equations in Lyra's manifold is as follows:

$$G_{ij}+\frac{3}{2}\psi_i\psi_j-\frac{3}{4}g_{ij}\psi_k\psi^k =-\frac{8\pi(T_{ij}+\bar{T}_{ij})}{\phi c^4}-\frac{\omega}{\phi^2}\left(\phi_{,i}\phi_{,j}-\frac{1}{2}g_{ij}\phi_{,k}\phi^k\right) -\frac{1}{\phi}(\phi_{,i,j}-g_{ij}\,\square\,\phi) \tag{9}$$

Where

$$\square\,\phi=\phi_{;i}^{,i}=\frac{8\pi(T+\bar{T})}{(3+2\omega)c^2} \tag{10}$$

where G_{ij},ψ^i,ω and ϕ are Einstein's curvature tensor, displacement vector field of Lyra's geometry, BransDicke coupling constant and scalar field respectively. Also the time like constant displacement vector is read as $\psi_i=(\beta,0,0,0)$.

T_{ij} and $\bar{T}_{ij}$ are energy momentum tensors for matter and holographic dark energy, respectively. Which are defined as

$$T_{ij}=\rho_m u_i u_j$$
$$\bar{T}_{ij}=(\rho_f+p_f)u_iu_j+g_{ij}p_f \tag{11}$$

Here ρ_m and ρ_h are the energy densities of matter and Fractional holographic dark energy and p_B is the pressure of holographic dark energy.

Also, the energy conservation equation is

$$(T_{ij}+\bar{T}_{ij})_{;j}=0, \tag{12}$$

The field equations (9)for the metric (8) with the help of (11) and (11), can be written as

$$2\frac{\ddot{B}}{B}+\frac{\dot{B}^2}{B^2}+\frac{3}{4}\beta^2=-\frac{8\pi}{\phi}p_f+\frac{\omega}{2}\frac{\dot{\phi}^2}{\phi^2}+2\frac{\dot{\phi}}{\phi}\frac{\dot{B}}{B}+\frac{\ddot{\phi}}{\phi} \tag{13}$$

$$\frac{\ddot{A}}{A}+\frac{\ddot{B}}{B}+\frac{\dot{A}\dot{B}}{AB}+\frac{3}{4}\beta^2=-\frac{8\pi}{\phi}p_f+\frac{\omega}{2}\frac{\dot{\phi}^2}{\phi^2}+\frac{\dot{\phi}}{\phi}\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}\right)+\frac{\ddot{\phi}}{\phi} \tag{14}$$

$$2\frac{\dot{A}\dot{B}}{AB}+\frac{\dot{B}^2}{B^2}-\frac{3}{4}\beta^2=\frac{8\pi}{\phi}(\rho_m+\rho_f)-\frac{\omega}{2}\frac{\dot{\phi}^2}{\phi^2}+\frac{\dot{\phi}}{\phi}\left(\frac{\dot{A}}{A}+2\frac{\dot{B}}{B}\right) \tag{15}$$

$$\ddot{\phi}+\dot{\phi}\left(\frac{\dot{A}}{A}+2\frac{\dot{B}}{B}\right)=\frac{8\pi}{3}\eta(\rho_m+\rho_f-3p_f) \tag{16}$$

we can write the energy conservation Eq.*(12)*of the matter and dark energy as

$$\dot{\rho_m}+\left(\frac{\dot{A}}{A}+2\frac{\dot{B}}{B}\right)\rho_m+\dot{\rho_f}+\left(\frac{\dot{A}}{A}+2\frac{\dot{B}}{B}\right)(1+\omega_f)\rho_f=0, \tag{17}$$

where overhead dot stands for ordinary differentiation with respect to t.

The continuity equation of the matter is

$$\dot{\rho}_m+\left(\frac{\dot{A}}{A}+2\frac{\dot{B}}{B}\right)\rho_m=0 \tag{18}$$

The continuity equation of the holographic dark energy is

$$\dot{\rho}_f+\left(\frac{\dot{A}}{A}+2\frac{\dot{B}}{B}\right)(\rho_f+p_f)=0 \tag{19}$$

Using Eqs.(18), (19)and the barotropic equation of state$p_f=\omega_f\rho_f$, the equation of state fractional HDE parameter is obtained as

$$w_f=-1-\frac{(3\gamma-2)(2\alpha H\dot{H}+\beta\ddot{H})}{6\gamma H(\alpha H^2+\beta\dot{H})}. \tag{20}$$

Now the average scale factor and the volume of the universe are defined as

$$a(t)=(AB^2)^{\frac{1}{3}} \tag{21}$$

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The spatial volume

$$V = a^3(t) = AB^2 \tag{22}$$

The directional Hubble parameters and average Hubble parameter

$$H_x = \frac{\dot{A}}{A}, H_y = \frac{\dot{B}}{B} \tag{23}$$

and

$$H = \frac{1}{3}\left(\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right) \tag{24}$$

The dynamical scalar expansion θ and shear scalar σ^2 are

$$\theta = 3H$$

$$\sigma^2 = \frac{1}{2}\sigma^{ij}\sigma_{ij} = \frac{1}{2}\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right)^2 \tag{25}$$

The average anisotropic parameter Δ

$$\Delta = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{H_i - H}{H}\right)^2 \tag{26}$$

Here H_i represents the directional Hubble parameters $(i = 1,2,3)$

The deceleration parameter is

$$q = -1 + \frac{d}{dt}\left(\frac{1}{H}\right) \quad \text{or} \quad q = -\left[\frac{a\ddot{a}}{\dot{a}^2}\right] \tag{27}$$

3. Solution of the field equations:

Now Eqs. (13)-(16) are a system of five independent equations in six unknowns A, B, C, ϕ, p_m and p_f.Hence to find a determinate solution we use the following physically plausible conditions (the question of over determinacy is settled by satisfying the field equations).

(i)

$$T + \bar{T} = \rho_m + \rho_\lambda - 3p_\lambda = 0 \tag{28}$$

which physically corresponds the vanishing of trace of both matter and dark energy tensors. This is analogous to the disordered radiation condition of general relativity.

(ii) we consider the relation between H and a, which was proposed by Berman (1983)[31]

$$H = na^{-\frac{1}{n}} \tag{29}$$

Where $l > 0$ and $n \geq 0$ are constants.

From Eqns. (27) and(29) we obtain

$$q = -1 + \frac{1}{n} \tag{30}$$

Now, using Eq.(*29*) and Eq.(*30*), the solution of Eq. (*27*) gives the law of variation of the average scale factor of the form

$$a(t) = (t + c)^n, \ n \neq 0 \tag{31}$$

Using $a(t) = 1/(z + 1)$, we obtained the following time red-shift relation

$$t(z) = \frac{1}{nl(z+1)^n} \tag{32}$$

Now, from Eq.(13) and Eq. (14), we get

$$\frac{\tilde{B}}{B} - \frac{\ddot{A}}{A} + \frac{\dot{B}^2}{B^2} - \frac{\dot{A}\dot{B}}{AB} = 0 \tag{33}$$

On integration gives,

$$\frac{\dot{B}}{B} - \frac{\dot{A}}{A} = \frac{c_1}{AB^2} \tag{34}$$

Where c_1 is constant of integration.

Using Eq. (*21*) in Eq.(*34*) and integrating again, we get

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$$\frac{B}{A} = c_2 \exp\left(\int \frac{c_1}{a^3} dt\right)$$

The metric functions A and B in terms of average scale factor $a(t)$ are given by

$$A(t) = c_2^{\frac{-2}{3}} a \exp\left(-\frac{2c_1}{3}\int a^{-3} dt\right)$$
$$B(t) = c_2^{\frac{1}{3}} a \exp\left(\frac{c_1}{3}\int a^{-3} dt\right) \tag{35}$$

Now using Eq.(31)in Eq.(35), we get

$$A(t) = c_2^{\frac{-2}{3}} (nlt)^{\frac{1}{n}} \exp\left(-\frac{2c_1}{3l(n-3)}(nlt)^{\frac{(n-3)}{n}}\right)$$
$$B(t) = c_2^{\frac{1}{3}} (nlt)^{\frac{1}{n}} \exp\left(\frac{c_1}{3l(n-3)}(nlt)^{\frac{(n-3)}{n}}\right) \tag{36}$$

where $n \neq 3$

$$A(z) = c_2^{-\frac{2}{3}} (z+1) \exp\left(-\frac{2c_1}{9l}(z+1)^{-3}\right)$$
$$B(z) = c_2^{\frac{1}{3}} (z+1) \exp\left(\frac{c_1}{9l}(z+1)^{-3}\right) \tag{37}$$

4. Kinematical parameters of the model

In this section, we compute the following kinematical parameters of the model (8), which play a significant role inthe discussion of the cosmological model of the universe.

The spatial volume of the metric is

$$V = a^3(t) = (nlt)^{\frac{-1}{n}} \tag{38}$$

The directional and average Hubble parameter

$$H_x = \frac{\dot{A}}{A} = \frac{1}{(nt)} - \frac{2c_1}{3}(nlt)^{\frac{-3}{n}}$$
$$H_y = \frac{\dot{B}}{B} = \frac{1}{(nt)} + \frac{c_1}{3}(nlt)^{\frac{-3}{n}} \tag{39}$$

and

$$H = (nt)^{-1} \tag{40}$$

The dynamical scalar expansion θ and shear scalar σ^2 are

$$\theta = 3(nt)^{-1} \tag{41}$$
$$\sigma^2 = c_1^2 (nlt)^{\frac{-6}{n}} \tag{42}$$

The average anisotropic parameter Δ

$$\Delta = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{H_i - H}{H}\right)^2 = \frac{2c_1^2}{l^2}(nlt)^{\frac{(2n-1)}{n}} \tag{43}$$

Applying the conservation condition for the left-hand side of Eq. (9), we get

$$\beta\left(\dot{\beta} + \beta\left[\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right]\right) = 0 \tag{44}$$

From Eq.(44)by integrating, we have

$$\beta = \frac{\beta_0}{AB^2} = \beta_0 (nlt)^{\frac{1}{n}} \tag{45}$$

From Eqns.(16)and(28), the scalar field ϕ satisfies the equation

$$\phi = \frac{\phi_0}{l(n-3)}(nlt)^{\frac{(n-3)}{n}} \tag{46}$$

Using Eq. (40) in Eq. (35), the pressure of FHDE is given by

$$p_f = \frac{(1-n)}{n^2 t^2} + \frac{c_1 (nlt)^{-\frac{2}{n}}}{9n^2 l^2} \tag{47}$$

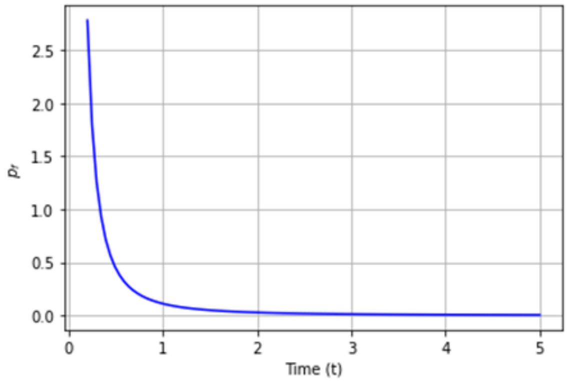

Fig (1) : Plot of pressure (p_f) of FHDE vs time (t).

Using Eqs. (43), (44) and (49) in Eq. (13), the energy density of matter is

$$\rho_m = \frac{3}{(nt)^2} - \frac{3c_1^2}{9}(nt)^{-\frac{6}{n}} - \frac{3}{4}\beta_0 (nlt)^{\frac{1}{n}} - 3\left(\frac{\alpha}{n^2 t^2} - \frac{\beta}{nt^2}\right)^{\frac{3\gamma-2}{2\gamma}} \tag{48}$$

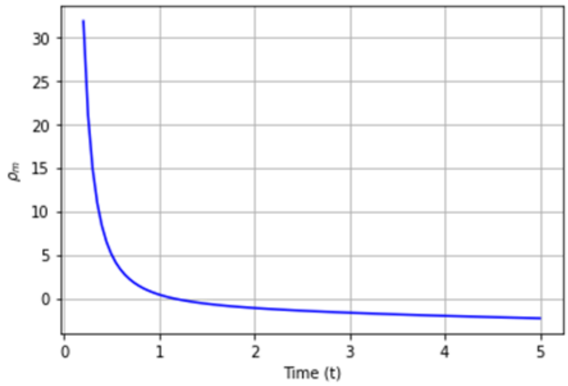

Fig (2): Plot of energy density (ρ_m) of matter vs red-shift (z)

Using Eq. (40) in Eq. **Error! Reference source not found.**, the energy density of FHDE is given by

$$\rho_f = 3\left(\frac{\alpha}{n^2 t^2} - \frac{\beta}{nt^2}\right)^{\frac{3\gamma-2}{2\gamma}}. \tag{49}$$

Eq. (50) shows that the ρ_f of fractional HDE model in Granda-Oliveros cut-off increases uniformly with increasing red-shift at all times.

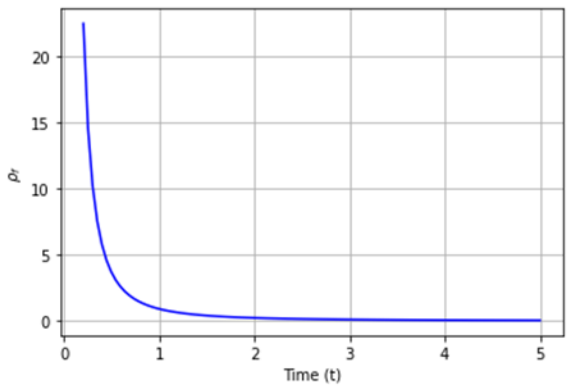

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Fig (3) : Plot of energy density (ρ_f) of Fractional holographic dark energy vs red-shift (z).

The Figs. (2) and (3) represent the plots of energy density (ρ_m) of matter and FHDE with the Granda-Oliveroscutoff decreases respectively. It is observed that both ρ_m and ρ_f are positive and decrease as universe evolves.

Using Eqs. (43), (44) and (49) in Eq. (15), the EoS parameter for FHDE is given by

$$w_f = -1 - \frac{n(3\gamma - 2)(\beta n - \alpha)}{3\gamma(\alpha - \beta n)} \tag{50}$$

From Eq. (50) shows the ω_f of fractional HDE model in Granda-Oliveros cut-off gives -1 it is typically depend upon the vales of α and β associated with dark energy or a cosmological constant, which leads to the accelerated expansion of the universe

Matter density parameter Ω_m and the holographic dark energy parameter Ω_B are given by

$$\Omega_m = \frac{\rho_m}{3H^2} \text{ and } \Omega_f = \frac{\rho_f}{3H^2} \tag{51}$$

Using Eqns.(40), (47), (48) and (51) we get the overall density parameter as

$$\Omega = \Omega_m + \Omega_f = 1 - \frac{c_1^2}{9}(nt)^{2-\frac{6}{n}} - \frac{1}{4}\beta_0 l^{\frac{1}{n}}(nt)^{\frac{1}{n}+2} \tag{52}$$

From Eq. (52)it can be seen that by increasing the value of z , the total energy density parameter of Fractional HDE model in Granda-Oliveros cut-off decreases below 1, indicating an open universe.

Quintessence Scalar Field Model

To establish the correspondence between the holographic dark energy with quintessence dark energy models, we compare the EoS and the dark energy density for the corresponding models of dark energy.

The action for the quintessence scalar ϕ is given by

$$S = \int d^4x\sqrt{-g}\left[-\frac{1}{2}g^{ij}\,\partial_i\phi\,\partial_j\phi - V(\phi)\right]$$

The energy density and pressure for the quintessence scalar field are respectively represented as:

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{53}$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{54}$$

where $V(\phi)$ is the potential energy.

The equation of state (EoS) for the scalar field is given by

$$\omega_\phi = \frac{\dot{\phi}^2 - 2V(\phi)}{\dot{\phi}^2 + 2V(\phi)} \tag{55}$$

For the accelerated expansion of the universe, the equation of state parameter for quintessence must be less than -1.

From eqn. (50), one can write

$$-1 - \frac{n(3\gamma - 2)(\beta n - \alpha)}{3\gamma(\alpha - \beta n)} = \frac{\dot{\phi}^2 - 2V(\phi)}{\dot{\phi}^2 + 2V(\phi)} \tag{56}$$

Also, comparing eqn. (7) and eqn. (50), we have

$$3(\alpha H^2 + \beta\dot{H})^{\frac{3\gamma-2}{2\gamma}} = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{57}$$

From eqn. (52) and eqn. (53), the kinetic energy term and the quintessence potential in power-law form can be obtained as follows:

$$\dot{\phi}^2 = 3\left(\frac{n(3\gamma - 2)(\beta n - \alpha)}{3\gamma(\alpha - \beta n)}\right)\left(\frac{\alpha}{n^2t^2} - \frac{\beta}{nt^2}\right)^{\frac{3\gamma-2}{2\gamma}}$$

$$\phi - \phi_0 = \left(\frac{n(3\gamma - 2)(\beta n - \alpha)}{3\gamma(\alpha - \beta n)}\right)^{\frac{1}{2}} \ln\left(\frac{t + c}{t_0 + c}\right)$$

Using eqn. (55) in eqn. (56), we obtained the potential in the exponential form

$$V(\phi) = \frac{-n(3\gamma-2)^2}{\gamma^2} \cdot \left(\frac{\alpha}{n^2t^2} - \frac{\beta}{nt^2}\right)^{\frac{3\gamma-2}{2\gamma}-1} \cdot \left(\frac{\beta}{nt^3} - \frac{\alpha}{n^2t^3}\right)$$

For $n > 1$, the potential $V(\phi)$ typically decreases with time, indicating an accelerated expansion of the universe. This means we can connect Fractional holographic dark energy with the quintessence scalar field. Essentially, we can describe holographic dark energy using the framework of quintessence, which is a type of dark energy model driven by a scalar field.

Conclusion

We have studied the LRS Bianchi type-I cosmological model within the framework of Fractional Holographic Dark Energy (FHDE) in the context of Brans-DickeTheory and Lyra Geometry. We observed that at the initial time, the metric potentials $A(t)$ and $B(t)$ are both zero. As time progresses, these potentials evolve, with their behavior dependent on the value of the parameter : for $n < 3$, they approach zero, and for $n > 3$, they tend towards infinity. Graphically, the cosmic scale factor (or average scale factor) is seen to increase with time. Specifically, as time t increases, . We observe that the spatial volume V is zero at $T = 0$. At this epoch, the physical parameters $H, \theta, \sigma, \rho_m, \rho_f$ and p_f are all infinite. Therefore the model has a point singularity at $T = 0$. As time increases, the spatial volume increases and the physical parameters decrease and ultimately tend to zero as $T \to \infty$.Therefore the ratio $\frac{\sigma}{\theta}$ does not tend to zero as $T \to \infty$. Therefore the shear scalar does not tend to zero faster than the expansion at late time. The anisotropy parameter Δ is zero at the initial time, implying that the model is isotropic at$t = 0$. It is observed that the holographic dark energy density decreases with the evolution of the universe and the average density parameter approaches to constant at late time, so the universe becomes flat which is supported to recent observations

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